Kernels and Convergence

Based on work with Tino Franz

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Contents

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The Ingredients

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- Particles with
  - position $\mathbf{x}_j(t) = P_t(jh)$, $j \in \mathbb{Z}^d$,
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  - mass $m_j = h^d \rho_0(jh)$.
- General assumption: Finite mass $\sum m_j < \infty$

The Simplest SPH Approximation

The approximation of a function $f : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is given by

$$f^\epsilon(\mathbf{x}, t) := \sum_{j \in \mathbb{Z}^d} f(\mathbf{x}_j(t), t) \frac{m_j}{\rho_j(t)} \Phi_\epsilon(\mathbf{x} - \mathbf{x}_j(t)).$$

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- More sophisticated approximations are possible.

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Remarks

- Approximation depends on ϵ and h.
- Derivatives of f are approximated by derivatives of f^ϵ.
- More sophisticated approximations are possible.
- Often: The exact particle trajectories are unknown.

Compressible Euler Equations

Velocity density, pressure: $\mathbf{u}, \rho, p$

$$
\begin{aligned}
\dot{\mathbf{x}}_j(t) &= \mathbf{u}_j(t), \\
\dot{\mathbf{u}}_j(t) &= -\frac{1}{\rho_j(t)}\nabla p(\mathbf{x}_j(t), t), \\
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Approximation:

$$\begin{aligned}
\dot{\mathbf{x}}_j^\epsilon(t) &= \mathbf{u}_j^\epsilon(t), \\
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\dot{\rho}_j^\epsilon(t) &= -\rho_j^\epsilon(t) \nabla \cdot \mathbf{u}^\epsilon(\mathbf{x}_j^\epsilon(t), t).
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Additional Assumptions/Concepts

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- Often, ρ is directly approximated by

$$\rho^\epsilon(\mathbf{x}, t) = \sum_{j \in \mathbb{Z}^d} m_j \Phi_\epsilon(\mathbf{x} - \mathbf{x}_j^\epsilon(t)).$$

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Results So Far

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$$\lim_{\epsilon\to 0}\lim_{h\to 0}\{\|\rho^\epsilon-\rho\|+\|\mathbf{u}^\epsilon-\mathbf{u}\|\}=0.$$

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  - Based upon work by Oehlschläger.
- Oehlschläger (1991):
  - Showed weak convergence for $p=\rho^2/2$.
  - Specific, technical assumptions on the kernel.

Oehlschläger's Result

- Convolution kernel

$$\Phi_\epsilon = \Phi_\epsilon^r * \Phi_\epsilon^r = \int_{\mathbb{R}^d} \Phi_\epsilon^r(\mathbf{y})\Phi_\epsilon^r(\cdot - \mathbf{y})d\mathbf{y}.$$

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$$\rho^\epsilon(\mathbf{x}, t) = \sum_{k\in\mathbb{Z}^d} m_k \Phi_\epsilon(\mathbf{x} - \mathbf{x}_k^\epsilon(t)).$$

- $p = \rho^2/2$, i.e. ODE system

$$\begin{aligned} \dot{\mathbf{x}}_j^\epsilon(t) &= \mathbf{u}_j^\epsilon(t), \\ \dot{\mathbf{u}}_j^\epsilon(t) &= -\nabla\rho^\epsilon(\mathbf{x}_j^\epsilon(t), t). \end{aligned}$$

Oehlschläger's Result

Energy:

$$Q(t) := \sum_{k\in\mathbb{Z}^d} m_k \|\mathbf{u}_k^\epsilon(t) - \mathbf{u}(\mathbf{x}_k^\epsilon(t), t)\|_2^2 + \int_{\mathbb{R}^d} \left| \sum_{k\in\mathbb{Z}^d} m_k \Phi_\epsilon^r(\mathbf{x} - \mathbf{x}_k^\epsilon(t)) - \rho(\mathbf{x}, t)\right|^2 d\mathbf{x}$$

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Theorem (Oehlschläger, 1991)

If the kernel Φ^r satisfies the approximation condition of order $\ell = \lfloor d/2 \rfloor + 1$ and if the solutions $\mathbf{u}, \rho$ are sufficiently smooth then

$$|Q(t)| \le C\left[Q(0)+\epsilon\right], \qquad t\in[0,T].$$

Consequences

- $Q(0)$ depends on h and ϵ. Consequently,

$$Q(t) \le C\left[\frac{h^{2n+d}}{\epsilon^{2n+d}} + \epsilon\right],$$

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- The result does not guarantee pointwise convergence, as $m_j = h^d \rho(jh)$ only gives

$$\|\mathbf{u}_j^\epsilon(t) - \mathbf{u}(\mathbf{x}_j^\epsilon(t), t)\|_2^2 \le m_j^{-1} Q(t) \le C \left[\frac{h^{2n}}{\epsilon^{2n+d}} + \frac{\epsilon}{h^d} \right].$$

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- The kernel conditions are formulated for Φ^r and not Φ.

Improved Result

Theorem (Franz/W.)

If the kernel Φ^r satisfies the moment condition of order m and the approximation condition of order $\ell > d/2$ and if the solutions $\mathbf{u}, \rho$ are sufficiently smooth then

$$\begin{aligned} Q(t) &\le C\left[Q(0) + \epsilon^{\min\{m,2\ell-d\}}\right] \\ &\le C\left[\frac{h^{2n+d}}{\epsilon^{2n+d}} + \epsilon^{\min\{m,2\ell-d\}}\right]. \end{aligned}$$

Point-Wise Convergence

Assume $m = 2\ell - d > d + d^2/(2n)$ and $\epsilon = h^{\frac{2n+d}{2n+d+m}}$. Then,

$$\begin{aligned} \|\mathbf{u}_j^\epsilon(t) - \mathbf{u}(\mathbf{x}_j^\epsilon(t), t)\|_2^2 &\leq m_j^{-1} Q(t) \leq C \left[\frac{h^{2n}}{\epsilon^{2n+d}} + \frac{\epsilon^m}{h^d} \right] \\ &= C \frac{\epsilon^m}{h^d} = C h^{\frac{2nm-2nd-d^2}{2n+d+m}} \end{aligned}$$

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Corollary

Under the above assumptions, we have compact convergence

$$\|\mathbf{x}_j(t) - \mathbf{x}_j^\epsilon(t)\|_2 + \|\mathbf{u}_j(t) - \mathbf{u}_j^\epsilon(t)\|_2 \to 0.$$

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The Moment Condition

In general: $\Phi : \mathbb{R}^d \to \mathbb{R}$ with $\int \Phi(\mathbf{x}) d\mathbf{x} = 1$.

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Definition

The kernel Φ satisfies the moment condition of order $m \in \mathbb{N}$ if it satisfies

$$\int_{\mathbb{R}^d} \mathbf{x}^\alpha \Phi(\mathbf{x}) d\mathbf{x} = 0$$

for $\alpha \in \mathbb{N}_0^d$ with $1 \leq |\alpha| < m$ and

$$\int_{\mathbb{R}^d} \|\mathbf{x}\|_2^m \Phi(\mathbf{x}) d\mathbf{x} < \infty.$$

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Lemma

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- *If* Φ *is non-negative then* Φ *is at most of moment order* 2.

Theorem

Let $1 \le p \le \infty$ *and let* Φ *be a kernel of order* m. *Then, there is a constant* $C > 0$ *such that for* $f \in W_p^m(\mathbb{R}^d)$,

$$\|f - f * \Phi_\epsilon\|_{L_p(\mathbb{R}^d)} \le C\epsilon^m \|f\|_{W_p^m(\mathbb{R}^d)}.$$

Moment Condition and Convolution Roots

Recall

$$\Phi(\mathbf{x}) = \Phi^r * \Phi^r(\mathbf{x}) = \int_{\mathbb{R}^d} \Phi^r(\mathbf{y})\Phi^r(\mathbf{x}-\mathbf{y})d\mathbf{y}.$$

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Theorem

Let Φ^r be even and $\Phi = \Phi^r * \Phi^r$. Then, Φ satisfies the moment condition of order m if and only if Φ^r satisfies the moment condition of order m.

Moreover, we have $\Phi_\epsilon = \Phi^r_\epsilon * \Phi^r_\epsilon$ for all $\epsilon > 0$.

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Fourier transform:

$$\widehat{\Phi}(\omega) = \int_{\mathbb{R}^d} \Phi(\mathbf{x})e^{-i\mathbf{x}^T\omega}d\mathbf{x}.$$

Oehlschläger's Approximation Condition

Definition

Let Φ^r be an integrable, even kernel with integral one. For $\alpha \in \mathbb{N}_0^d$ let $p_\alpha(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The kernel Φ^r satisfies the approximation property of order ℓ if there is a constant $C > 0$ such that

$$|(p_\alpha \partial_j \Phi^r)^\wedge(\omega)| \leq C|\widehat{\Phi}^r(\omega)|, \qquad \omega \in \mathbb{R}^d \tag{1}$$

holds for all $\alpha \in \mathbb{N}_0^d$ with $1 \leq |\alpha| \leq \ell$ and $1 \leq j \leq d$, and such that

$$p_\alpha \partial_j \Phi^r \in L_2(\mathbb{R}^d) \tag{2}$$

holds for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = \ell + 1$ and $1 \leq j \leq d$.

Alternative Views

- Condition (1) is equivalent to

$$|D^\alpha[\omega_j \widehat{\Phi}^r(\boldsymbol{\omega})]| \le C|\widehat{\Phi}^r(\boldsymbol{\omega})|, \qquad \boldsymbol{\omega} \in \mathbb{R}^d, \quad 1 \le |\alpha| \le \ell. \qquad (3)$$

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- Condition (2) is satisfied if (1) also holds for $|\alpha| = \ell + 1$ and if $\Phi^r \in L_2(\mathbb{R}^d)$.

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General Approaches

Two possibilities:

- Choose Φ^r, compute $\Phi = \Phi^r * \Phi^r$.
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Goal: Find conditions on Φ so that Φ^r exists and satisfies the approximation and moment conditions.

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- $\Phi = \phi(\|\cdot\|_2)$ radial with $\phi : [0, \infty) \to \mathbb{R}$.

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Theorem

If $\sigma > d/2$ then for each $0 < \nu < \sigma - d/2$:

$$\Phi^r \in W_1^{\sigma-d/2-\nu}(\mathbb{R}^d) \cap W_2^{\sigma-d/2-\nu}(\mathbb{R}^d).$$

Approximation Property

Theorem

If $f = \mathcal{F}_d \phi$ satisfies

$$\left| \frac{d^j}{dt^j} [tf(t)] \right| \le Cf(t), \qquad t \in [0,\infty), \quad 1 \le j \le \ell$$

and, with a $\mu > 0$,

$$\left| \frac{d^{\ell+1}}{dt^{\ell+1}} [tf(t)] \right| \le C(1+|t|)^{-(d+\mu)/2}, \qquad t \ge 0$$

then

$$\Phi^r = \mathcal{F}_d^{-1} f^{1/2}$$

satisfies the approximation property of order ℓ.

Example: Compactly Supported Functions

Let $\nu > 0$ and $f : [0, \infty) \to \mathbb{R}$. Let

$$\begin{aligned} \phi_\nu(r) &:= (1-r)_+^\nu, \\ \mathcal{I}f(r) &:= \int_r^\infty t f(t) dt, \qquad r \geq 0. \end{aligned}$$

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- $\ell \geq 1$: *all satisfy* $0 < \mathcal{F}_d \phi_{d,k,\ell}(r) \leq C(1+r)^{-d-2k-1}$.

Example $d = 3$

k	ℓ	$\phi(r)$	order
1	0	$(1-r)_+^4(4r+1)$	-
	1	$(1-r)_+^5(5r+1)$	-
	2	$(1-r)_+^6(6r+1)$	1
	3	$(1-r)_+^7(7r+1)$	2
2	0	$(1-r)_+^6(35r^2+18r+3)$	-
	1	$(1-r)_+^7(16r^2+7r+1)$	-
	2	$(1-r)_+^8(21r^2+8r+1)$	1
	3	$(1-r)_+^9(80r^2+27r+3)$	2

Example $d = 3$

k	ℓ	$\phi(r)$	order
1	0	$(1-r)_+^4(4r+1)$	-
	1	$(1-r)_+^5(5r+1)$	-
	2	$(1-r)_+^6(6r+1)$	1
	3	$(1-r)_+^7(7r+1)$	2
2	0	$(1-r)_+^6(35r^2+18r+3)$	-
	1	$(1-r)_+^7(16r^2+7r+1)$	-
	2	$(1-r)_+^8(21r^2+8r+1)$	1
	3	$(1-r)_+^9(80r^2+27r+3)$	2

Theorem

The function $\phi_{d,k,\ell}$ satisfies the approximation property of order ℓ and the moment condition of order $m = 2$. Hence,

$$Q(t) \le C[Q(0) + \epsilon^{\min\{2,2\ell-3\}}].$$

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- Determine $\lambda_0, \dots, \lambda_{k-1}$ so that moment conditions are satisfied.

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Theorem

If ψ is such that $\mathcal{F}_d\psi > 0$ and $\mathcal{F}_d\psi$ is monotonically decreasing then $\mathcal{F}_d\phi > 0$ and $\Phi = \phi(\|\cdot\|_2)$ satisfies the moment condition of order $m = 2k$.

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- *The previously defined functions $\psi = \phi_{d,k,\ell}$ have $\mathcal{F}_d\phi_{d,k,\ell}$ monotonically decreasing for $\ell \geq 1$.*

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Remark

- *The previously defined functions $\psi = \phi_{d,k,\ell}$ have $\mathcal{F}_d\phi_{d,k,\ell}$ monotonically decreasing for $\ell \geq 1$.*
- *Though ϕ is not non-negative anymore, it still often has a convolution root.*

Conclusions

Summary:

- Proof of convergence for Euler's equations with $p = \rho^2/2$.
- Explicit kernels with compact support.
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To Do:

- More general pressure-density relations.
- Boundary conditons.
- Numerical tests for new kernels.